ON THE ROOTS OF FUNCTIONS CONNECTED BY A LINEAR RECURRENT RELATION OF THE SECOND ORDER.*

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AT the close of his first memoir on homogeneous linear differential equations of the second order in the first volume of Liouville's Journal, Sturm remarks that results analogous to those obtained for the differential equation were first discovered by him while considering solutions of a homogeneous equation in finite differences of the form,

$$L_n y_{n+1} + M_n y_n + N_n y_{n-1} = 0, \tag{1}$$

when L, M, and N are functions, subject to certain restrictions, of the positive integral index n and a variable parameter, and that it was while prosecuting this inquiry that he discovered his famous theorem concerning the isolation of the roots of a polynomial. By passing from the discrete index n to the continuous variable x, Sturm first arrived at the theorems for the differential equation in the paper referred to. This earlier method of dealing with the problems solved by Sturm was perhaps regarded by him as merely heuristic, or at least as less elegant than the methods finally employed, and his earlier researches were never published.

Characterizing a sequence of real functions $y_0, y_1, \ldots y_n$ as *Sturmian*† when the difference in the number of variations of sign in the sequence for two particular real values of the argument is equal to the number of real roots of y_n in the interval delimited by them, it is certain that Sturm determined at least one simple sufficient condition to which a sequence must conform in order that it may be *Sturmian*, and that he investigated the inverse problem of constructing such sequences by means of a recurrent relation of type (1). Looked at from the standpoint of the recurrent relation, solutions of (1) enjoying the *Sturmian* property may be called *Sturmian solutions*; undoubtedly Sturm discovered many properties of such solutions.

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* Read before the American Mathematical Society, February, 1901, under a different title.

† This adjective is here used in a more general sense than in the ensuing sections. *Cf.* the definitions there given.

It is our purpose in the first five sections to reproduce in part these unpublished theorems and in the last section to show how, by means of the Cauchy-Lipschitz theorem for the existence of solutions of a differential equation, it is possible to establish rigorously the analogous theorems, so far as they exist, for the homogeneous linear differential equation of the second order. As has already been stated, in this case the index n is replaced by the independent variable and the variable parameter either figures in the differential equation itself, or in the arbitrary parameters of integration, or in both.

In conclusion it may be said that it is highly probable that Sturm considered a somewhat more general difference equation than (1′) of §3 and so arrived at criteria of a more general character. This point, however, I hope to consider elsewhere.

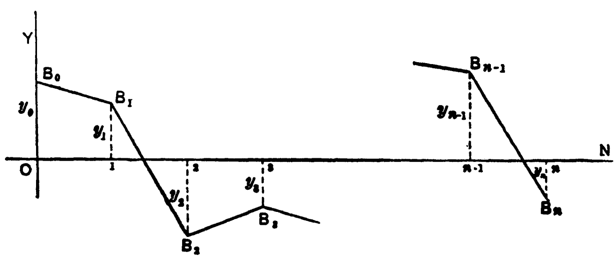
1. Sturmian Sequences in General. We begin by considering a series of functions

$$y_n(x), \quad y_{n-1}(x), \; . \; . \; . \; . \; . \; y_0(x)$$

of the real variable x, real, single valued, and analytic in an interval $x_0 \leqq x \leqq x_1$.*

A. It is further supposed that for no point of this interval is $y_0(x)$ equal to zero.

It will be convenient to employ the following graph in considering the



above sequence. The ordinate $i\,B_i$ representing y_i in magnitude and sign, the abscissa $O\,i$ may conveniently be taken equal to the index i. In the graph just described, the intersections of the broken line $B_0\,B_1\; . \; . \; . \; . \; B_n$ with the N-axis, which we shall call *v-points*, correspond to the *variations of sign* in the sequence $y_0, y_1, \; . \; . \; . \; y_n$.

If now we suppose the parameter x to vary continuously, this broken line will undergo a continuous deformation, the point B_0 remaining always above

* Throughout the rest of this paper we shall suppose that the variable x is confined to such an interval.

the origin, while the v-points are displaced in a manner which will depend on the particular functions $y_1, \ldots y_n$. It is evident that at least one variation will be lost or gained when y_n changes sign. The vanishing of an intermediate y can cause the loss or gain of a variation only when $B_0 B_1 \ldots B_n$ is deformed in such a manner that, when a given y vanishes, the adjacent y's have the same sign, or one of the adjacent y's is zero. In order that the number of variations of sign in the sequence $y_0, y_1, \ldots y_n$ should change as x varies from x_0 to x_1 *only* when y_n changes sign, it will be *sufficient*, if we impose the condition:

B. No two consecutive y's can vanish for the same value of x, and when any intermediate y vanishes the adjacent y's have opposite signs.

A series of functions of the kind specified, *i. e.* fulfilling Conditions A and B, are said to form a *generalized Sturmian* sequence.*

Let us now consider what further restrictions must be imposed on our sequence in order that it should lose a variation each time y_n vanishes. Let x' be a value of x for which $y_n = 0$. The sequence will lose a variation as x increases through the value x' when and only when y_n and y_{n-1} have opposite signs for values of x a little smaller than x' and the same sign for values a little larger, *i. e.* when

$$(2) \qquad \frac{y_n(x' - \Delta x)}{y_{n-1}(x' - \Delta x)} < 0 < \frac{y_n(x' + \Delta x)}{y_{n-1}(x' + \Delta x)} \qquad \left(\begin{array}{l}\Delta x = \text{a sufficiently small} \\ \text{positive quantity.}\end{array}\right)$$

From (2) we see that y_n/y_{n-1} is increasing with x at the point x', and therefore we have as a consequence of (2):

$$(3) \qquad \left[\frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right)\right]_{x=x'} \geqq 0.$$

On the other hand, if a variation is to be gained:

$$\left[\frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right)\right]_{x=x'} \leqq 0.$$

We shall treat explicitly only the case when variations are lost as x increases.

It is to be noted that condition (2) is more general than (3) and that we cannot pass back to (2) except when the inequality sign holds. It is thus a

* *Cf.* Netto *Algebra*, vol. 1, p. 238.

sufficient, but not a necessary, condition for the loss of a variation that*

$$(4) \qquad \frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right) = \frac{y_n'}{y_{n-1}} > 0 \qquad \text{when } y_n = 0.$$

When this condition is fulfilled as well as conditions A and B, the sequence is called a *Sturmian* sequence.† It should be noted that condition (4) makes it impossible for y_n to have a multiple root in the interval (x_0, x_1) considered.

Starting with a real polynomial which has no multiple roots in (x_0, x_1) and choosing $y_{n-1} \equiv y_n'$, Sturm's modified algorithm of the Greatest Common Divisor leads to a sequence possessing these properties, the ordinary Sturm's Functions of text books on the Theory of Equations.

2. The Difference Equation. If y_n denote a function of the integral index n, where we shall always suppose n to denote a positive integer or zero, y_n is said to satisfy a *homogeneous linear recurrent relation of the second order* if we have for all values of n,

$$(1) \qquad L_n y_{n+1} + M_n y_n + N_n y_{n-1} = 0,$$

where L_n, M_n, and N_n denote functions of the index n. We will assume in what follows that none of the functions L_n is zero.

If we write

$$\begin{aligned} y_{n+1} - y_n &= \Delta y_n, \\ y_{n+2} - y_{n+1} &= \Delta y_{n+1}, \end{aligned}$$

and

$$\Delta y_{n+1} - \Delta y_n = \Delta^2 y_n,$$

relation (1) may be written in the form of a difference equation

$$L_n' \Delta^2 y_n + M_n' \Delta y_n + N_n' y_n = 0,$$

a form entirely analogous to that of the homogeneous linear differential equation of the second order. We shall however employ this equation only in the recurrent form.

If to y_0 and y_1 be assigned any values we choose,‡ all subsequent y's are uniquely determined by means of (1), since we have assumed that L_n does not vanish; so that (1) may be said to have ∞^2 solutions.

If $y_n^{(1)}$ and $y_n^{(2)}$ denote any two solutions, then, $C_1 y_n^{(1)} + C_2 y_n^{(2)}$ is also a

* If multiple roots are excluded (4) is of course as general as (2).

† *Cf.* Weber, *Algebra*, vol. 1, p. 272 (1st ed.).

‡ This is analogous to the determination of a solution of a differential equation of the second order by its value and the value of its derivative at a given point.

solution.* $y_n^{(1)}$ and $y_n^{(2)}$ are said to be linearly dependent if a relation of the form

$$C_1 y_i^{(1)} + C_2 y_i^{(2)} = 0, \qquad i = 0, 1, 2 \ldots n$$

exist between them, where the C's denote constants.

We will now establish the theorem:

The necessary and sufficient condition for the linear dependence of two solutions $y_n^{(1)}$ and $y^{(2)}$ is

(5)
$$y_0^{(1)} y_1^{(2)} - y_0^{(2)} y_1^{(1)} = 0.$$

For if $y_n^{(1)}$ and $y_n^{(2)}$ are linearly dependent, we have:

(6)
$$\begin{aligned} C_1 y_0^{(1)} + C_2 y_0^{(2)} &= 0, \\ C_1 y_1^{(1)} + C_2 y_1^{(2)} &= 0, \end{aligned}$$

where C_1 and C_2 are not both zero, and therefore (5) is fulfilled.

Conversely, if (5) is fulfilled, there exist two constants C_1 and C_2 not both zero which satisfy (6). Now if we have:

$$\begin{aligned} C_1 y_k^{(1)} + C_2 y_k^{(2)} &= 0, \\ C_1 y_{k-1}^{(1)} + C_2 y_{k-1}^{(2)} &= 0, \end{aligned}$$

then by equation (1) we have at once,

$$C_1 y_{k+1}^{(1)} + C_2 y_{k+1}^{(2)} = 0.$$

Therefore starting from (6) we see that:

$$C_1 y_n^{(1)} + C_2 y_n^{(2)} = 0 \qquad (n = 2, 3, 4, \ldots .).$$

If then $y_n^{(1)}$ and $y_n^{(2)}$ denote any two linearly independent particular solutions and y_n the general solution, and if we determine C_1 and C_2 by means of the equations,

$$y_0 = C_1 y_0^{(1)} + C_2 y_0^{(2)},$$

$$y_1 = C_1 y_1^{(1)} + C_2 y_1^{(2)},$$

then
$$y_n = C_1 y_n^{(1)} + C_2 y_n^{(2)} \qquad (n = 0, 1, 2, 3, \ldots .)$$

i. e. any solution can be expressed linearly in terms of any two particular linearly independent solutions.

In the next section we shall be mainly concerned with a recurrent relation of the form,

(1′)
$$y_{n+1} + G_n y_n + y_{n-1} = 0.$$

* Analogous to the two arbitrary constants in the general solution of a differential equation of the 2nd order.

Equation (1) may be reduced to this form by the change of dependent variable

$$(7) \qquad y_n = e_n z_n.$$

Thus (1) becomes

$$L_n e_{n+1} z_{n+1} + M_n e_n z_n + N_n e_{n-1} z_{n-1} = 0.$$

If now we determine e_n by the recurrent relation

$$L_n e_{n+1} = N_n e_{n-1}$$

we get an equation of the desired form, where e_0 and e_1 may be chosen at pleasure, subject merely to the restriction that they do not vanish, and

$$e_{2n} = \frac{N_{2n-1} N_{2n-3} \cdot \cdot \cdot \cdot N_1}{L_{2n-1} L_{2n-3} \cdot \cdot \cdot \cdot L_1} e_0, \qquad e_{2n+1} = \frac{N_{2n} N_{2n-2} \cdot \cdot \cdot \cdot N_2}{L_{2n} L_{2n-2} \cdot \cdot \cdot \cdot L_2} e_1,$$

$$G_n = \frac{M_n}{L_n} \frac{e_n}{e_{n+1}}.$$

3. Sturmian Sequences satisfying a Recurrent Relation of Type (1). We shall now consider a sequence of functions y_n $(n = 0, 1, \ldots n)$ satisfying a relation of the form,

$$L_n y_{n+1} + M_n y_n + N_n y_{n-1} = 0.$$

in which L, M, and N are functions of the index n and are real, single valued, and analytic functions of a real variable x in an interval $x_0 \leqq x \leqq x_1$. If, as we shall assume to be the case, $L_n(x)$ does not vanish in this interval, y_2 and all y's of higher index will be determined as real single valued analytic functions of x in this interval as soon as y_0 and y_1 are assigned as arbitrary real analytic functions of x.

Let us now choose y_0 so that it will not vanish in the interval (x_0, x_1) and seek to determine what conditions may be imposed on L, M, N, y_0, and y_1, so that the sequence $y_0, y_1, \ldots y_n$ shall form a Sturmian sequence. To ensure that no two consecutive y's shall vanish for the same value of x and that, when any y vanishes, the adjacent y's shall have opposite signs, it will be sufficient if we suppose that *not only L but also N does not vanish in the interval* (x_0, x_1) *and that these functions have like signs*, *i. e.* that LN is always positive. If these conditions are satisfied, the y's form a generalized Sturmian sequence.

To determine a sufficient condition that

$$\left[\frac{d}{dx}\left(\frac{y_n}{y_{n-1}}\right)\right]_{x=x'} > 0,$$

when x' denotes a root of y_n in the interval (x_0, x_1), we shall suppose (1) reduced to the form (1′) by the transformation (7) of §2 and consider the equation as given in the form,

$$(1') \qquad y_{n+1} + G_n y_n + y_{n-1} = 0,$$

when evidently the condition $LN > 0$ is fulfilled.* The relation between z_n and y_n (§2) shows that, under the restrictions placed on L and N, z and y vanish together.

Let x take the positive increment Δx, and let us write $y_i(x + \Delta x) = \bar{y}_i$, etc. Then (1′) becomes

$$\bar{y}_{n+1} + \bar{G}_n \bar{y}_n + \bar{y}_{n-1} = 0.$$

So that, when $y_n \bar{y}_n$ is not zero, we have, since $y_0(x) \neq 0$:

$$\Delta \frac{y_{n+1}}{y_n} = \frac{\bar{y}_{n+1}}{\bar{y}_n} - \frac{y_{n+1}}{y_n} = \frac{1}{\bar{y}_n y_n}\left[(-\bar{G}_n + G_n)\,\bar{y}_n y_n + \bar{y}_n y_{n-1} - y_n \bar{y}_{n-1}\right]$$

$$(8) \qquad = \frac{1}{\bar{y}_n y_n}\left[\sum_1^n (-\bar{G}_m + G_m)\,\bar{y}_m y_m + \bar{y}_0 y_0\left(\frac{\bar{y}_1}{\bar{y}_0} - \frac{y_1}{y_0}\right)\right].$$

Since all the y's are continuous, we can, for a given value of x, take Δx so small that

$$\bar{y}_m y_m \geqq 0, \qquad m = 0, 1, 2, \ldots n.$$

Let us now impose the restrictions:

$$\left.\begin{array}{ll} 1^\circ & \bar{G}_m \leqq G_m \\ & \quad m = 1, 2 \ldots n. \\ 2^\circ & \bar{y}_1/\bar{y}_0 \geqq y_1/y_0 \end{array}\right\} \qquad x_0 \leqq x \leqq x_1$$

where, if the equality sign holds in 2°, we require either that there be two successive relations 1° where it does not hold, or that it does not hold in the last relation 1°.

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* It may be remarked in passing that this transformation is unnecessary if N/L is independent of x.

Under these circumstances a quantity δ can be found such that, if $y_n(x) \neq 0$,

$$\Delta \frac{y_{n+1}}{y_n} > 0 \qquad \text{when } \Delta x < \delta.$$

Thus y_{n+1}/y_n will then increase continually with x from $-\infty$ to $+\infty$ in every interval of (x_0, x_1) delimited by two consecutive roots of $y_n(x)$. Since the y's are analytic functions of x, the limits

$$\lim_{\Delta x = 0} \frac{\Delta\left(\frac{y_{n+1}}{y_n}\right)}{\Delta x}, \qquad \lim_{\Delta x = 0} \frac{\Delta\left(\frac{y_1}{y_0}\right)}{\Delta x}$$

exist and we have,

$$(9) \qquad \left(\frac{y_{n+1}}{y_n}\right)' = \frac{1}{y_n^2}\left[\sum_1 - G'_m y_m^2 + y_0^2 \left(\frac{y_1}{y_0}\right)'\right] \qquad y_n \neq 0.$$

The conditions

$$1^\circ \quad G'_m(x) \leqq 0, \qquad\qquad m = 1, 2, \ldots n$$

$$2^\circ \quad \left(\frac{y_1}{y_0}\right) \geqq 0,$$

the restrictions above stated as to the simultaneous presence of equalities in 1° and 2° holding, yield at once the inequality

$$\left(\frac{y_{n+1}}{y_n}\right)' > C > 0, \qquad y_n \neq 0,$$

when x is supposed to lie in an interval included within an interval delimited by two consecutive roots of y_n. Thus, if conditions 1° and 2° are fulfilled, the functions $y_0, y_1, y \ldots y_n$ form a Sturmian sequence. We may call such solutions *Sturmian solutions.*

Let us now consider the case in which G_n and $y\ /y_0$ are real analytic functions of several parameters $x_1, x_2, \ldots$ in the intervals

$$x_0^{(i)} < x_i < x_1^{i)}, \qquad i = 1, 2, \ldots.$$

We will assume that these functions satisfy the conditions

$$1^\circ \quad \frac{\partial}{\partial x_i} G_m \leqq 0, \qquad\qquad m = 1, 2, \ldots n$$

$$2^\circ \quad \frac{\partial}{\partial x_i}\left(\frac{y_1}{y_0}\right) \geqq 0. \qquad\qquad i = 1, 2, \ldots.$$

Let us first consider the case in which there are only two parameters x_1, x_2, and

ask ourselves how the roots of y_{n+1}, regarded as a function of x_1, vary with x_2. It is clear that these roots are continuous (in fact analytic) functions of x_2 which, since they are never multiple roots, can cease to be real only by disappearing at one end of our interval. The question is whether they increase or decrease with x_2. To settle this question we notice that, x_1, x_2 being a pair of values for which $y_n \neq 0$, the function y_{n+1}/y_n must, according to what we have proved above, increase when x_1 or x_2 or both increase. If then x_1', x_2' is a pair of values for which y_{n+1} vanishes, and if starting from these values, we wish to change x_1, x_2 so that y_{n+1} retains the value zero, we must increase one of them and decrease the other; *i. e.* if we regard y_{n+1} as a function of x_1, its roots decrease as x_2 increases.

Coming back now to the case in which there are more than two parameters, we will assign to all but two of them fixed values. Then by applying the result just obtained we get the theorem:

If $y_n(x_1, x_2 \ldots)$ be regarded as a function of the variable x_i, the roots of y_n will decrease with the increase of the other parameters.

4. Theorems concerning Sturmian Solutions. We have seen that, if y_n denote a Sturmian solution in a certain interval, then y_n has no multiple roots in this interval and that y_n/y_{n-1} increases continually from $-\infty$ to $+\infty$ in any interval delimited by two consecutive roots of y_{n-1}. Similarly, it is seen at once that $(y_n + H_n y_{n-1})/y_{n-1}$ will increase continually from $-\infty$ to $+\infty$ in this same interval provided that

$$\frac{d}{dx} H_n \geqq 0.$$

A similar extension can of course be given to the last theorem of §3. From the foregoing we can now deduce the following theorems.

THEOREM 1. *A variation will be lost in the sequence $y_0, y_1, \ldots y_n$ whenever y_n vanishes.*

THEOREM 2. *The roots of y_n and y_{n-1} separate each other*, or more generally *the roots of $y_n + H_n y_{n-1}$ and y_{n-1} separate each other provided that $H_n' \geqq 0$.*

For, since y_n/y_{n-1} increases continually from $-\infty$ to $+\infty$ in any interval delimited by two consecutive roots of y_{n-1}, the function y_n, which has no multiple roots, will vanish once and only once when x increases through this interval.

Theorem 3. *If $y_n^{(1)}$ and $y_n^{(2)}$ denote any two linearly independent solutions of* (1′), *then if* $\left(\dfrac{y_1^{(1)}}{y_0^{(1)}}\right)' > 0$ *and* $\left(\dfrac{y_1^{(2)}}{y_0^{(2)}}\right)' > 0$, *the roots of $y_n^{(1)}$ and $y_n^{(2)}$ will separate each other.*

From (1′) we have

$$y_n^{(1)} y_{n-1}^{(2)} - y_n^{(2)} y_{n-1}^{(1)} \equiv y_1^{(1)} y_0^{(2)} - y_1^{(2)} y_0^{(1)} \equiv C(x)$$

where, by the hypothesis that $y_n^{(1)}$ and $y_n^{(2)}$ are linearly independent, C must be a function of x which does not vanish in the interval (x_0, x_1). Further, since $C(x)$ is continuous, and thus cannot change sign, we may suppose $C(x)$ always positive. Let $\bar{x}_2$ and x_2 denote two consecutive roots of $y_n^{(2)}$. By Theorem 2 this interval contains a single root of $y_{n-1}^{(2)}$; hence, since

$$y_n^{(1)}(x_2)\, y_{n-1}^{(2)}(x_2) = C(x_2) > 0$$

and

$$y_n^{(1)}(\bar{x}_2)\, y_{n-1}^{(2)}(\bar{x}_2) = C(\bar{x}_2) > 0,$$

it follows that $y_n^{(1)}$ vanishes at least once in the interval determined by two consecutive roots of $y_n^{(2)}$. Similarly, $y_n^{(2)}$ vanishes at least once in an interval determined by two consecutive roots of $y_n^{(1)}$, and hence the theorem.

Let $y_n^{(1)}$ and $y_n^{(2)}$ be any two linearly independent solutions, and let us denote by y_n the solution

$$y_n = z y_n^{(1)} + y^{(2)}$$

where z is a parameter. Then we have

$$\frac{y_n}{y_{n-1}} = \frac{z y_n^{(1)} + y_n^{(2)}}{z y_{n-1}^{(1)} + y_{n-1}^{(2)}}$$

and

$$\frac{\partial}{\partial z}\left(\frac{y_n}{y_{n-1}}\right) = \frac{y_n^{(1)} y_{n-1}^{(2)} - y_n^{(2)} y_{n-1}^{(1)}}{(z y_{n-1}^{(1)} + y_{n-1}^{(2)})^2} = \frac{y_0^{(1)} y_0^{(2)}}{(z y_{n-1}^{(1)} + y_{n-1}^{(2)})^2}\left(\frac{y_1^{(1)}}{y_0^{(1)}} - \frac{y_1^{(2)}}{y_0^{(2)}}\right).$$

Thus y_n/y_{n-1} increases continually with z between any two consecutive roots of y_{n-1}, if $y_1^{(1)}/y_0^{(1)} > y_1^{(2)}/y_0^{(2)}$. If z decrease from a very large value to a very small one, y_n/y_{n-1}, starting from a value as near $y_n^{(1)}/y_{n-1}^{(1)}$ as we please, will approach $y_n^{(2)}/y_{n-1}^{(2)}$ as its limit, and the v-points of $y_n \ldots y_0$ will, starting from coincidence with those of $y_n^{(1)} \ldots y_0^{(1)}$, all move in the same sense into coincidence with those of $y_n^{(2)} \ldots y_0^{(2)}$. Since y_n always remains linearly independent of $y_n^{(1)}$ and $y_n^{(2)}$, a v-point of $y_n \cdots y_0$ could not come into coincidence with one of $y_n^{(2)} \ldots . y_0^{(2)}$. If $y_1^{(1)}/y_0^{(1)} < y_1^{(2)}/y_0^{(2)}$ we can reason in the same way, and thus we get

THEOREM 4. *If $y_n^{(1)}$ and $y_n^{(2)}$ denote any two linearly independent solutions of (4) such that $y_0^{(1)}$ and $y_0^{(2)}$ do not vanish, no restrictions as to continually increasing or decreasing being imposed on $G_n(x)$, the v-points of the series $y_n^{(1)} \ldots y_0^{(1)}$ and of $y_n^{(2)} \ldots y_0^{(2)}$ will alternate, i. e. the number of variations in the two sequences will differ at most by unity.*

Theorems 3 and 4 are analogous to well known theorems of Sturm for the linear differential equation of the second order. Theorems 1 and 2 seem to possess no analogon for the differential equation.

The analogon of Sturm's *Theorem of Oscillation* may be stated as follows.

THEOREM 5. *If $G_n(x)$ continually increases with x and is such that $G_n(x_0) \geqq L^2$ and $G_n(x_1) \leqq -M^2$ $(n = 0, 1, 2, \ldots)$, and if L^2 and M^2 are sufficiently large, there is one and only one value of $x (x_0 \leqq x \leqq x_1)$ such that*

1° *The number of variations in the sequence $y_n, y_{n-1}, \ldots y_1$ is equal to an assigned integer r (r not greater than $n - 1$) and such that*

2°
$$\frac{y_1(x)}{y_0(x)} = a$$

$$\frac{y_n(x)}{y_{n-1}(x)} = a'$$

a and a' being any assigned real constants.

The proof is immediate. We note that, if in the equation

$$\bar{y}_{n+1} + C\bar{y}_n + \bar{y}_{n-1} = 0$$

we take C as a sufficiently large positive constant, the sequence $\bar{y}_n, \bar{y}_{n-1}, \ldots \bar{y}_1$ will present $n - 1$ variations of sign and if C is a negative constant of sufficiently large absolute value, this sequence will present no variations of sign. Thus if L^2 is sufficiently large, the sequence $y_1, y_2, \ldots y_n$ will present $n - 1$ variations of sign. As x increases from x_0 towards x_1 G will decrease and a variation will be lost every time y_n vanishes; further since the roots of y_{n-1} separate those of y_n and in an interval delimited by two successive roots of y_{n-1}, y_n/y_{n-1} increases continually from $-\infty$ to $+\infty$ with the increase of x, y_n/y_{n-1} passes in every such interval through any given value, as β, once and only once. This proves the theorem.

5. Some Applications. As examples of sequences which fall at once under the conditions of the preceding sections we will cite first the

Zonal Harmonics:

$$P_0(x),\ \ P_1(x), \ldots\ P_n(x)\ ; \quad \text{where } P_0 = 1 \text{ and } P_1 = x,$$

while the recurrent relation is

$$(n+1)P_{n+1} = (2n+1)\,xP_n - nP_{n-1}.$$

Since L_n and N_n do not depend on x, the conclusions of §3 are valid and we have, the interval (x_0, x_1) extending from $-\infty$ to $+\infty$,

1° The sequence $P_n \ldots P_0$ is *Sturmian*;

2° The roots of P_n and P_{n-1} separate each other;

3° P_n has no multiple roots.

From 1° it follows that all the roots of P_n are real.*

Lommel's Function: $R^{n,m}(x)$, defined as follows,

$$J_n(x) = R^{n,m-1}(x)\,J_{n+m-1}(x) - R^{n,m-2}(x)\,J_{n+m}(x),$$

where the J's denote Bessel functions, affords a second example.

Here $R^{n,0}(x) = 1$ and $R^{n,1}(x) = 2(n+1)/x$, the recurrent relation being†

$$R^{n,m+1}(x) = 2\frac{(n+m+1)}{x}R^{n,m}(x) - R^{n,m-1}(x), \tag{10}$$

so that, if $n > -1$, the conditions of §3‡ are fulfilled and $R^{n,0}, R^{n,1}, \ldots R^{n,m}$ form a Sturmian sequence. The roots of $R^{n,m}$ and $R^{n,m-1}$ are all real and alternate and increase with the parameter n.

Even if $n < -1$, if $n + m < 0$, the conditions of §3 and the conclusions from them hold.

The conditions at the end of §3 also have place for (10) if $n + m < 0$ and x be changed to $-x$, so that, *if n be increased, the roots of $R^{n,m}(-x) = 0$ decrease; i. e.* the roots of $R^{n,m}(x) = 0$ increase with n.

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* These theorems may be obtained by making use of the relation

$$(x^2-1)P_n' = n\,x\,P_n - n\,P_{n-1}$$

which shows that, if $|x| < 1$, $P_n' P_{n-1} > 0$, when $P_n = 0$ (conditions on p. [57]). *Cf.* Weber's *Algebra*, vol. 1, p. 277.

† *Cf.* Graf and Gubler's *Einleitung in die Theorie der Besselschen Functionen*, vol. 2, p. 102.

‡ The inequalities 1° and 2° of p. 62 are here reversed, but Theorems 1, . . . 5 still subsist if we replace the word *lost* in Theorem 1 by *gained* and reverse the inequalities in Theorem 3.

As a final example we may consider the partial numerators and denominators of the convergents of the continued fraction

$$F(x) = a_1 x + \beta_1 - \frac{\gamma}{a_2 x + \beta_2 -} \frac{\gamma_2}{a_3 x + \beta_3 -}$$

when the a's are all positive or all negative and the γ's are positive functions of the index n. Denoting the partial numerators and denominators by y_n and $\bar{y}_n$ respectively we have

$$\begin{aligned} y_n &= (a_n x + \beta_n) y_{n-1} - \gamma_n y_{n-2}, \\ \bar{y}_n &= (a_n x + \beta_n) \bar{y}_{n-1} - \gamma_n \bar{y}_{n-2}, \end{aligned}$$

$$y_0 = 1, \qquad y_1 = a_1 x + \beta,$$
$$\bar{y}_0 = 0, \qquad \bar{y}_1 = 1, \qquad \bar{y}_2 = a_2 x + \beta_2.$$

Thus the conditions of §3 are fulfilled and $y_n, y_{n-1}, \ldots . y_0$ (or $\bar{y}_n, \bar{y}_{n-1}, \ldots . \bar{y}_1$) form a Sturmian sequence, all the roots of $y_n(\bar{y}_n)$ are real and are separated by those of $y_{n-1}(\bar{y}_{n-1})$.* From the relation

$$y_n \bar{y}_{n-1} - \bar{y}_n y_{n-1} = \prod_2^n \gamma_n$$

we also infer that the roots of y_n and $\bar{y}_n$ separate each other, *i. e.* the poles and zeros of $y_n/\bar{y}_n$ alternate.

By §3 we conclude that the absolute values of the roots of $y_n(x) = 0$ $(\bar{y}_n(x) = 0)$ decrease with $a_r (r \leqq n)$ if the a's are positive, and increase with the a's if the a's are negative.

The foregoing is the simplest type of continued fraction which satisfies the conditions of §§3 and 4.

The general type suggested by §3 is

$$\phi_0(x) - \frac{\gamma_1}{\phi_1(x) -} \frac{\gamma_2}{\phi_2(x) -} \ldots$$

where $\phi_n'(x) > 0$.

6. The Limiting Case of the Differential Equation. To show that under certain restrictions, to be hereafter given, the difference equation goes over into a differential equation as its limiting form we shall need the following

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* This result has been established by Sylvester, *Philosophical Transactions*, vol. 14, part I.

Lemma. If $f(x)$ denote a single valued function of the real variable x possessing finite first and second derivatives in the interval (x_0, x_1) *then**

$$\lim_{\delta=0} \frac{f(x+2\delta) - 2f(x+\delta) + f(x)}{\delta^2} = f''(x), \qquad x_0 < x < x_1.$$

Consider now the difference equation,

$$f_{n+2} + G_n f_{n+1} + f_n = 0 \tag{11}$$

and let

$$f_n = f(a + n\delta), \qquad n\delta = x - a,$$

then (11) may be written

$$\frac{f_{n+2} - 2f_{n+1} + f_n}{\delta^2} = -\left(\frac{2+G_n}{\delta^2}\right) f_{n+1} \tag{12}$$

If then we suppose that

$$\lim_{\delta=0} \frac{2+G_n}{\delta^2} = G(x)$$

equation (11) will go over into the differential equation:

$$f''(x) = -G(x) f(x).$$

Conversely, if we have a differential equation in the form

$$y'' = G(x)\, y \tag{13}$$

the Cauchy-Lipschitz existence theorem for solutions of differential equations† shows that, if $G(x)$ is continuous (not necessarily analytic) in the neighbor hood of a, we can find a solution of (2) having the arbitrary boundary values

$$y(a) = c_1, \qquad y'(a) = c_2 \tag{14}$$

by regarding (13) as the limiting form of a difference equation. In the proof as usually presented (13) is replaced by the simultaneous pair of equations of the first order

$$p' = -G(x)\, y$$
$$y' = p$$

which are regarded as the limit of the difference equations

$$\begin{aligned} p_{n+1} - p_n &= -G(a + n\delta)\, y_{n+1}\delta, \\ y_{n+1} - y_n &= p_n \delta. \end{aligned} \tag{15}$$

* *Cf.*, for example, Harnack, *Elemente der Differential- und Integralrechnung*, p. 56.

† See for instance Picard, *Traité d'analyse*, vol. 2, p. 291.

Here we must let

$$y_0 = c_1, \qquad p_0 = c_2.$$

By means of the two equations of (14), all y's and p's of higher order can be computed, and the Cauchy proof shows that

$$\lim_{n=\infty} y_n = y(x), \qquad \lim_{n=\infty} p_n = y'(x)$$

where $y(x)$ is a solution of (13).

The upper limit for $n\delta = |x - a|$ postulated in the classic proof is in general much too small. Picard and Painlevé* have shown that the Cauchy solution will be convergent throughout the whole interval $b_0 b_1$ $(b_0 < a < b_1)$, in which $y(x)$ as well as $G(x)$ are continuous and that the convergence is uniform over every tract within this interval. Moreover, since we are dealing with a linear equation, the interval throughout which y is continuous will be at least as great as the interval throughout which G is continuous.†

It is well known‡ that no solution of (13) can have a multiple root at a point of the interval just considered. From this it follows that there cannot be two distinct solutions of (13) satisfying conditions (14), since their difference would have a multiple root at a. On the other hand it is clear that a solution of (2) which is not identically zero cannot have an infinite number of roots in a finite interval within and at the extremities of which G is continuous. For, at the points of condensation of such roots, y' as well as y would be zero.

If we now suppose that $y(a)$, $y'(a)$, and $G(x)$ are continuous functions of a parameter λ $(\lambda_0 < \lambda < \lambda_1)$ then y_n and p_n are continuous functions of this parameter and, by the uniformity of the convergence, their limits $y(x)$ and $y'(x)$ are continuous functions of (x, λ).

From the method in which y_n and p_n must be computed by means of (14) it is evident that the result will be the same if we eliminate p between the two equations of (14), thus getting

$$(15) \qquad y_{n+2} - 2y_{n+1} + y_n = - G(a + n\delta)\, y_{n+1}\, \delta^2.$$

If now we plot the ordinates y_m, $m = 0, 1, 2, \ldots n$, as in §1, using for the

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* Picard, *Comptes Rendus*, vol. 128 (1899), p. 1363.
Painlevé, *Bull. de la Soc. Math. de France*, vol. 27 (1899), p. 150.

† *Cf.* Peano, *Math. Annalen*, vol. 32 (1888), p. 450.

‡ *Cf.* Sturm, *l. c.* p. 109.

corresponding abscissas $m\delta$, $m = 0, 1, \ldots n$, we shall get a broken line which, if δ be taken sufficiently small, will lie between the two curves

$$\left.\begin{aligned} y &= y(x) + \epsilon \\ y &= y(x) - \epsilon \end{aligned}\right\} \quad (\epsilon \text{ arbitrarily small})$$

and

where x is supposed to be any point of an interval lying within the interval in which $y(x)$ is continuous.

Equation (15) belongs to the type of difference equations considered in §3, and it was then shown (*cf.* (8)) that if either y_1/y_0 or $G(\lambda)$ increases with λ, the other not decreasing, or both are functions of the parameter λ which continually increase as λ increases $y_n(\lambda)/y_{n-1}(\lambda)$ increases continually with λ in any interval delimited by two successive roots of $y_n(\lambda) = 0$, and that the same is true of $\dfrac{y_n(\lambda) - y_{n-1}(\lambda)}{y_{n-1}(\lambda)\,\delta}$ so that we have the fundamental theorem :

Under the restrictions imposed on $y'(a, \lambda)/y(a, \lambda)$ and $G(x, \lambda)$, namely, that either or both shall continually increase with λ, and neither decrease, the ratio $y'(x, \lambda)/y(x, \lambda)$ will be a continually increasing function of λ in any interval delimited by two successive roots of $y(x, \lambda) = 0$.

In the same way it can be shown that the roots of $y(x, \lambda) = 0$ *will continually increase with λ.*

The Oscillation Theorem for the differential equation can be arrived at in the same way by regarding it as a limiting form of the analogous theorem for the difference equation.

The equation (8) of §3 becomes, if we pass to the limit,

$$y(x, \lambda)\, y'_x(x, \lambda_1) - y(x, \lambda_1)\, y'_x(x, \lambda) =$$
$$\int_a^x [\,G(x,\lambda) - G(x,\lambda_1)\,]\, y(x,\lambda)\, y(x,\lambda_1)\, dx + y'_x(a,\lambda_1)\, y(a, \lambda) - y'_x(a,\lambda)\, y(a,\lambda_1).$$

This fundamental equation, which can be at once obtained from the differential equation itself, is the starting point of Sturm and Bôcher* in establishing the Comparison Theorems from which the other Sturmian Theorems are deduced.

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* The Theorems of Oscillation of Sturm and Klein., *Bull. Am. Math. Soc.*, ser. 2, vol. 4 (1898), p. 295.